

## Fourier Series

### A) Orthogonal Representations of signals:

A set of signals  $\phi_i(t)$ ,  $i=0, \pm 1, \pm 2, \dots$  is called orthogonal over an interval  $(a, b)$ , where  $a > -\infty$  and  $b \leq +\infty$ , if

$$\int_a^b \phi_l(t) \phi_k^*(t) dt = \begin{cases} E_k & \text{for } l=k \\ 0 & \text{for } l \neq k \end{cases} = E_k \delta(l-k)$$

Note that for  $l=k$ ,  $E_k$  is the energy of signal  $\phi_k(t)$  over interval  $(a, b)$ .

Note If  $E_k$  are all equal to 1, then the set is called Orthonormal.

An Orthogonal set  $\phi_i(t)$  can be made an orthonormal set when  $\psi_i(t) = \frac{\phi_i(t)}{\sqrt{E_i}}$ .

Examples the following sets are each orthogonal:

(1)  $\phi_k(t) = \sin(kt)$ ,  $k=1, 2, 3, \dots$  on the interval  $-\pi < t < \pi$

Because

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(kt) \sin(nt) dt &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(k-n)t dt \\ &\quad - \frac{1}{2} \int_{-\pi}^{\pi} \cos(k+n)t dt \\ &= \begin{cases} \pi & k=n \\ 0 & k \neq n \end{cases} \end{aligned}$$

(2)  $\phi_k(t) = e^{j\left(\frac{2\pi k}{T}\right)t}$  for  $k=0, \pm 1, \pm 2, \dots$   
over interval  $(0, T)$ .

Note that:  $\phi_k(t+T) = \phi_k(t)$ , i.e  $T$  is the period.

And  $\int_0^T \phi_n(t) \phi_k^*(t) dt = \begin{cases} T & \text{for } n=k \\ 0 & \text{for } n \neq k. \end{cases}$

and  $\frac{1}{\sqrt{T}} e^{j\left(\frac{2\pi k}{T}\right)t}$  for all integer  $k$  constitute an Orthonormal set over the interval  $0 < t < T$ .

### The Exponential Fourier Series representation of a Periodic Signal

\* Let  $x(t) = x(t+T)$ ,  $T > 0$  is fundamental period of  $x(t)$ .

Then

$$\begin{cases} x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2\pi n t}{T}}, & j = \sqrt{-1} \\ C_n = \frac{1}{T} \int_0^T x(t) e^{-j\frac{2\pi n t}{T}} dt & -\infty < n < \infty. \end{cases} \quad 0 < t < T.$$

Note:  $\frac{2\pi}{T} = 2\pi F_0 = \omega_0$  fundamental radian frequency.

\*  $C_n = |C_n| e^{j\phi_n} \Rightarrow$  Discrete function of frequency.

\* If  $x(t) = x(t+T)$  is real valued signal

then  $|C_n| = |C_{-n}|$  and  $\phi_n = -\phi_{-n}$  \* phase spectrum of  $x(t)$ .

Magnitude spectrum of  $x(t)$       Even Discrete function of frequency      Odd Discrete function of frequency.

Note

\* For any  $n = 0, \pm 1, \pm 2, \dots$ ,

the frequency is  $n\omega_0 = n 2\pi F_0 = n \frac{2\pi}{T}$

\*  $F_0 = \frac{1}{T}$  is the fundamental frequency in Hz (Hertz = cycle per second), or fundamental harmonic.

\*  $n F_0$  is the  $n$ th harmonic in Hz.

\* For real valued  $x(t) = x(t+T)$ ,  $x(t)$  can be written as (using  $|C_n| = |C_{-n}|$  and  $\phi_n = -\phi_{-n}$ ):

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right) \right]$$

where

$$a_0 = C_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = 2 \operatorname{Re}\{C_n\} = \frac{2}{T} \int_0^T x(t) \cos\left(\frac{2\pi n t}{T}\right) dt$$

$$b_n = -2 \operatorname{Im}\{C_n\} = \frac{2}{T} \int_0^T x(t) \sin\left(\frac{2\pi n t}{T}\right) dt$$

or :

$$x(t) = C_0 + \sum_{n=1}^{\infty} 2 |C_n| \cos\left(\frac{2\pi n t}{T} + \phi_n\right) .$$

\* Note If  $x(t) = x(t+T)$  is real and even then  
The Fourier Series Coefficients are real:

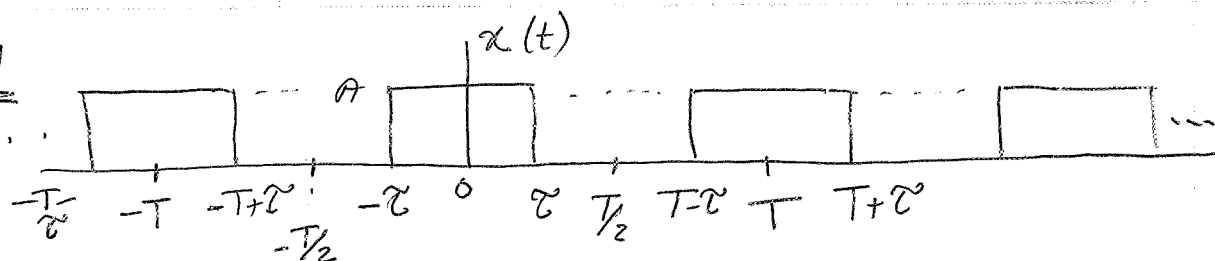
$C_n$  is real value.

Then  $|C_n| = |C_{-n}|$ .

\* If  $C_n$  is positive real value then  $\phi_n = 0$   
for any  $n$ .

\* If  $C_n$  is negative real value then  $\phi_n = +\pi$  or  
 $\phi_n = -\pi$ , for any  $n$ .

\* If  $C_n = 0$  for any  $n$ , then  $\phi_n$  is undefined.

Ex 1

Note  $x(t)$  is real, even, and periodic with period  $T$ .

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j \frac{2\pi n t}{T}} dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j \omega_0 n t} dt$$

where  $\omega_0 = \frac{2\pi}{T} = 2\pi F_0$ ,  $F_0 = \frac{1}{T}$

Now

$$C_n = \frac{1}{T} \int_{-\tau}^{\tau} A e^{-j \omega_0 n t} dt = \frac{A}{T} \int_{-\tau}^{\tau} e^{-j \omega_0 n t} dt =$$

$$= \frac{A}{T} \left\{ \frac{1}{-j \omega_0 n} e^{-j \omega_0 n t} \right|_{t=-\tau}^{t=\tau} \right\} =$$

$$= \frac{A}{T} \left( \frac{1}{-j \omega_0 n} \right) \left\{ e^{-j \omega_0 n \tau} - e^{j \omega_0 n \tau} \right\} =$$

$$= \frac{A}{T} \left( \frac{1}{-j \omega_0 n} \right) \left\{ -2j \sin(\omega_0 n \tau) \right\} =$$

$$= \frac{A}{T} \left( \frac{2}{\omega_0 n} \right) \sin(\omega_0 n \tau) = \left( \frac{2A\tau}{T} \right) \frac{\sin(\omega_0 n \tau)}{\omega_0 n \tau}$$

$$C_n = \frac{2A\tau}{T} \text{Sinc}(\omega_0 n \tau) = \frac{2A\tau}{T} \text{Sinc}\left(\frac{2\pi n \tau}{T}\right)$$

$C_n$  is real for all  $n$ .  $C_n$  is a Sinc function.

\* At  $n=0 \Rightarrow C_0 = \frac{2A\tau}{T}$

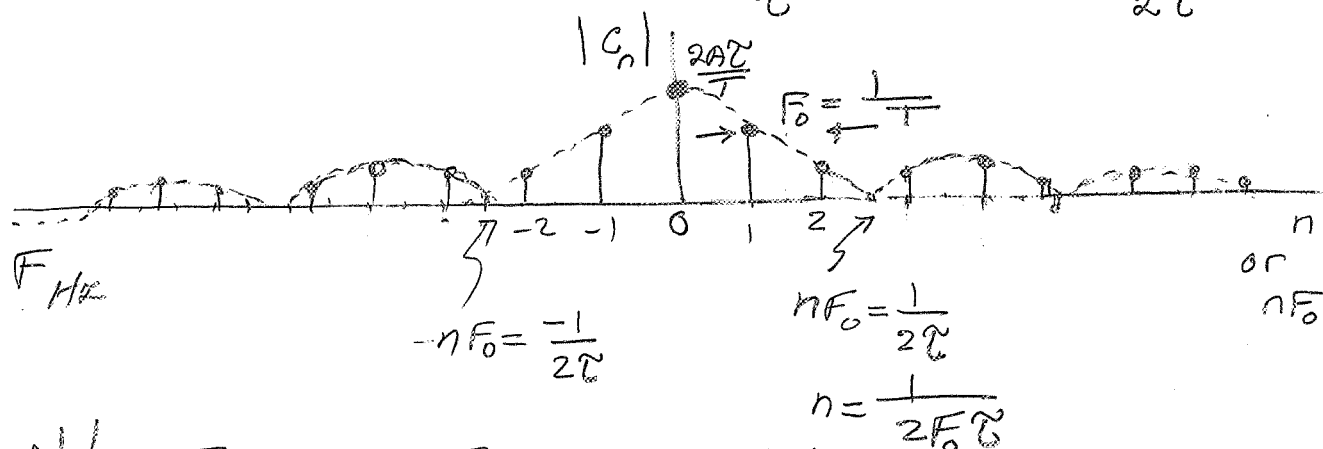
\* Zero crossing of  $C_n$ :

$$\omega_0 n\tau = k\pi \quad k = \pm 1, \pm 2, \pm 3, \dots$$

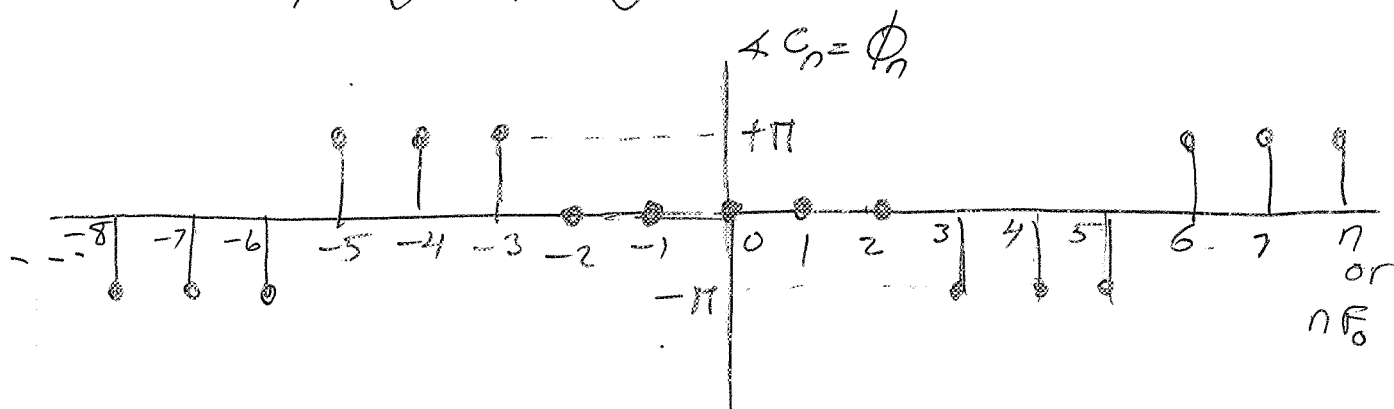
or  $\frac{2\pi}{T} n\tau = k\pi = 2\pi F_0 n\tau \Rightarrow nF_0 = \frac{k}{2\tau}$

For  $k = \pm 1 \Rightarrow nF_0 = \pm \frac{1}{2\tau}$  or  $nF_0 = \pm \frac{1}{2\tau}$

For  $k = \pm 2 \Rightarrow nF_0 = \pm \frac{2}{2\tau}$  or  $nF_0 = \pm \frac{2}{2\tau}$



Note Frequency spacing is  $F_0 = \frac{1}{\tau}$



Ex. 2

$$x(t) = 5 - 7 \cos\left(\frac{\pi}{3} t\right) + \sin\left(\frac{5\pi}{6} t\right),$$

To find  $T$  such that  $x(t) = x(t+T)$ :  $-\infty < t < +\infty$ .

$$\frac{\pi}{3} = \frac{2\pi}{T_c} \Rightarrow \frac{1}{T_c} = \frac{1}{6} \Rightarrow \boxed{T_c = 6} \text{ for } \cos\left(\frac{\pi}{3} t\right).$$

$$\frac{5\pi}{6} = \frac{2\pi}{T_s} \Rightarrow T_s = \frac{12}{5} \text{ for } \sin\left(\frac{5\pi}{6} t\right).$$

Now  $\frac{T_s}{T_c} = \frac{12/5}{6} = \frac{2}{5} \Rightarrow 2T_c = 5T_s = T$

$\boxed{T = 12}$

Now we can write  $x(t)$  as:

$$x(t) = 5 - \frac{7}{2} e^{j\frac{\pi}{3}t} - \frac{7}{2} e^{-j\frac{\pi}{3}t} + \frac{1}{2j} e^{j\frac{5\pi}{6}t} - \frac{1}{2j} e^{-j\frac{5\pi}{6}t}$$

and using  $T=12$ , we can write  $x(t)$  as:

$$x(t) = 5 - \frac{7}{2} e^{j\frac{2\pi}{12}t} - \frac{7}{2} e^{-j\frac{2\pi}{12}t} + \frac{1}{2j} e^{j\frac{2\pi \cdot 5}{12}t} - \frac{1}{2j} e^{-j\frac{2\pi \cdot 5}{12}t}$$

Now, we compare the above with:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2\pi}{T}nt} \quad \text{where } T=12.$$

Thus, we obtain:

$$|C_2| = \frac{7}{2},$$

$$C_0 = 5 \text{ and } \angle C_0 = 0, \quad C_2 = C_{-2}^* = -\frac{7}{2} \text{ and } \angle C_2 = +\pi$$

$$C_5 = C_{-5}^* = \frac{1}{2j} = -\frac{1}{2}j \text{ and any other } C_n = 0.$$

$$|C_5| = \frac{1}{2} \text{ and } \angle C_5 = -\frac{\pi}{2}.$$

Dirichlet Conditions

Conditions for  $x(t) = x(t+T)$  to have Convergent Fourier Series expansion:

- (1)  $\int_0^T |x(t)| dt < \infty$  integrable over one period
- (2)  $x(t)$  has finite number of maxima and minima over one period.
- (3)  $x(t)$  has finite number of discontinuities over one period.

The above are sufficient conditions.



## Properties of Fourier Series

(1) If  $x(t) = x(t+T)$  is real and even then  $C_n$  are real for all  $n$ , and

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) = x(-t)$$

$$a_0 = \frac{2}{T} \int_0^{T/2} x(t) dt, \quad a_n = \frac{4}{T} \int_0^{T/2} x(t) \cos\left(\frac{2\pi n t}{T}\right) dt$$

(2) If  $x(t) = x(t+T)$  is real and Odd then  $C_n$  are Imaginary.

And  $x(t) = -x(-t)$  where

$$x(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{T}\right),$$

$$b_n = \frac{4}{T} \int_0^{T/2} x(t) \sin\left(\frac{2\pi n t}{T}\right) dt$$

(3) Linearity

$$x(t+T) = x(t) \longrightarrow C_n$$

$$y(t+T) = y(t) \longrightarrow d_n$$

Then

$v(t+T) = v(t) = \alpha x(t) + \beta y(t)$  will give

$$v(t) \longrightarrow b_n = \alpha C_n + \beta d_n$$

where  $C_n$ ,  $d_n$  and  $b_n$  are respective Fourier Series Coefficients.

#### (4) Product of two signals

$$x(t+T) = x(t), \quad y(t+T) = y(t)$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{-jn\omega_0 t}, \quad y(t) = \sum_{m=-\infty}^{\infty} d_m e^{-jm\omega_0 t}$$

then

$$v(t) = x(t)y(t) = \sum_{k=-\infty}^{\infty} b_k e^{-jk\omega_0 t}$$

where

$$b_k = \sum_{\alpha=-\infty}^{\infty} C_{k-\alpha} \cdot d_{\alpha}$$

Convolution Sum  
of  
two sequences  $C_n$  and  $d_n$ .

#### (5) Convolution of two time signals

$$x(t) = x(t+T), \quad h(t) = h(t+T)$$

then  $y(t) = x(t) * h(t)$  over one period  $T$

$$y(t) = \frac{1}{T} \int_0^T x(\alpha) h(t-\alpha) d\alpha$$

$$x(t) \rightarrow C_n, \quad h(t) \rightarrow d_n, \quad \text{and} \quad y(t) \rightarrow b_n$$

where  $b_n = C_n \cdot d_n$  for all  $n$  and

$b_n$ ,  $C_n$ , and  $d_n$  are respective Fourier Series Coefficients of  $y(t)$ ,  $x(t)$ , and  $h(t)$ .

(6) Parseval's Theorem:

Given  $x(t) = x(t+T)$ ,  $y(t) = y(t+T)$  and

$$\left. \begin{array}{l} x(t) \rightarrow C_n \\ y(t) \rightarrow d_n \end{array} \right\} \begin{array}{l} \text{Fourier Series Coefficients} \\ \text{(i.e. Spectrum of } x(t) \\ \text{and spectrum of } y(t)) \end{array}$$

Now

$$P_x = \frac{1}{T} \int_0^T |x(t)|^2 dt$$

$$P_y = \frac{1}{T} \int_0^T |y(t)|^2 dt$$

Also:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\omega_0 n t}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$y(t) = \sum_{m=-\infty}^{\infty} d_m e^{j\omega_0 m t}$$

$$C_n = \frac{1}{T} \int_0^T x(t) e^{-j\omega_0 n t} dt$$

$$d_m = \frac{1}{T} \int_0^T y(t) e^{-j\omega_0 m t} dt$$

Now

$$P_x = \frac{1}{T} \int_0^T x(t) x^*(t) dt = \frac{1}{T} \int_0^T x(t) \left\{ \sum_{n=-\infty}^{\infty} C_n^* e^{-j\omega_0 n t} \right\} dt$$

$$P_x = \sum_{n=-\infty}^{\infty} C_n^* \left\{ \frac{1}{T} \int_0^T x(t) e^{-j\omega_0 n t} dt \right\}$$

$$P_x = \sum_{n=-\infty}^{\infty} C_n C_n^* = \sum_{n=-\infty}^{\infty} |C_n|^2 = \frac{1}{T} \int_0^T |x(t)|^2 dt$$

" Parseval's theorem "

And similarly for  $y(t)$

$$P_y = \frac{1}{T} \int_0^T |y(t)|^2 dt = \sum_{m=-\infty}^{\infty} |d_m|^2$$

Now

$$\text{If } h(t) = h(t+T) = x(t) * y(t)$$

Then

$$b_k = c_k d_k \quad \text{for } -\infty < k < \infty$$

where

$$b_k = \frac{1}{T} \int_0^T h(t) e^{-j\omega_0 t} dt$$

$$\text{Thus } |b_k|^2 = |c_k d_k|^2 = |c_k|^2 |d_k|^2$$

$$\text{and } P_h = \sum_{k=-\infty}^{\infty} |b_k|^2 = \sum_{k=-\infty}^{\infty} |c_k|^2 |d_k|^2$$

Note  $|c_n|^2$ ,  $|d_n|^2$ , and  $|b_n|^2$  are

called Power Density Spectrum of  $x(t)$ ,  $y(t)$ , and  $h(t)$ , respectively.

(7) Shift in Time

If  $x(t) = x(t+T) \rightarrow C_n$

then  $x(t) = x(t-\alpha) \rightarrow b_n = e^{-j\omega_n \alpha} C_n,$

Now notice that

$$0 < \alpha < T$$

$$|b_n| = |C_n| = |C_n|$$

$$\angle b_n = \angle C_n - \omega_n \alpha$$

That is, a shift in time of  $x(t)$  results in a linear phase shift for the phase spectrum of  $x(t)$ . No effect on the magnitude spectrum of  $x(t)$ .

--> See Page 206 - Oppenheim book

Note We will discuss response of a LTI system to a periodic signal, and also the Gibbs phenomenon later on in ch. 4.